UNIVERSITÉ DU LUXEMBOURG
ANALYSE 1
2015-2016

EXERCISE SHEET 6

6.1. Consider the hyperbolic functions

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

6.1.1. Prove that $\cosh^2(x) - \sinh^2(x) = 1$ for every $x \in \mathbb{R}$

6.1.2. Prove that $\cosh(x) = \cos(ix)$ and $\sinh(x) = -i\sin(ix)$

6.1.3. Compute their derivatives

6.1.4. Compute the Taylor series of $\cosh(x)$ and $\sinh(x)$.

6.2. We want to compute the Taylor series of $\arctan(x)$ for $x \to 0$ and $x \to +\infty$:

6.2.1. prove that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$;

6.2.2. deduce the Taylor series of $\arctan(x)$;

6.2.3. prove that $\arctan(x) + \arctan(\frac{1}{x}) = \frac{\pi}{2}$;

6.2.4. deduce the asympotic expansion (i.e. the Taylor expansion for $x \to +\infty$) of $\arctan(x)$.

6.3. Compute the Taylor expansion of order 4 for $x \to 0$ of the following functions

6.3.1. $f(x) = \cos(2x)$

6.3.2. $f(x) = \sqrt{1 - 2x^2}$

6.3.3. $f(x) = e^x \sin(2x)$

6.3.4. $f(x) = \log(\cos(x) + \sin(x))$

6.4. Compute the limit of the following functions for $x \to 0$

6.4.1. $f(x) = \frac{x^2 - 2 + 2\cos(x)}{x^4}$

6.4.2. $f(x) = \frac{\sqrt[3]{(1+x)} + \sqrt[3]{(1-x)} - 2}{1 - \cos(x) + x^4}$

6.4.3. $f(x) = \frac{\exp(x) + \cos(x) - 2 - x}{\arctan(x) - \sin(x)}$

6.4.4. $f(x) = \frac{\log(1 + \sin^2(x)) - x^2}{\sin^2(x^2) - x^5}$

6.4.5. $f(x) = \exp(x)(\log(e^x + 1) - x)$ (for $x \to +\infty$)

6.4.6. $f(x) = \left[1 - \left(\frac{\sin(x)}{x}\right)^{x^2}\right]\frac{1}{x^4}$

6.4.7. $f(x) = \frac{\arctan(x^5) - \arctan^5(x)}{x^7}$